ON SOCCER BALL

Marián Trenkler[1]

Catholic University in Ružomberok,
Faculty of Education, Department of Mathematics

Abstract. A soccer ball is stitched together from 32 pieces of leather, 12 of which have the shape of a regular pentagon and 20 of a regular hexagon. In the lecture we examine the number and the shape of faces from which a soccer ball can be made. A part of the contribution is devoted to the study of metric properties of regular convex polytopes and the groups of their symmetries.

Keywords. Soccer ball, combinatorial properties, symmetries.

There are times when the whole world watches 22 young men chasing a ball on a soccer field, trying to get it into the competitor's goal.

In this contribution we shall study this small ball (685 – 690 in diameter and weighing between 425 and 435 grams) which, from time to time, gives many of us a hard time.

In 1863 the Football Association (also known as simply the FA) which approved the first Laws of the Game was established. There was no description of the soccer ball in these laws though. In 1872 the laws were revised, and the shape, size and weight of the soccer ball were set. With only minor changes, these regulations are used up till now. For official competitions there are more strict rules, e.g. the different diameters of the ball may not differ more then by 1,5 percent.

All this and much more (including an uncountable number of pictures) can be found on the Internet, hence we do not continue in this direction. Instead, we shall deal with the combinatorial properties of the soccer ball.

Let us start with the properties of the soccer ball which was used on FIFA (International Federation of Association Football) tournaments in the years 1970–2002. We shall explore its combinatorial properties and its symmetries.

This, the standard and the most known soccer ball, is stitched together from pieces of leather in the shape of 12 regular pentagons and 20 regular hexagons, which we shall call *faces*. Pairs of faces meet along *edges* and triples of faces meet

[1]This research is supported by KEGA project-463-035KU-4/2010

in *vertices*. It is natural to ask whether there exist different balls consisting of the same number of pentagons and hexagons with the property that three faces meet in each vertex.

The Steinitz theorem ([1], page 33.) states that a graph is the edge graph of a polyhedron if and only if it is a simple planar graph which is 3-connected. The soccer ball can be considered as a polyhedron. Hence, according to the Stenitz theorem, instead of the ball it suffices to consider a simple 3-connected planar graph (a graph is said to be k-connected if there does not exist a set of $k-1$ vertices whose removal disconnects the graph).

Let us state some properties of convex polyhedrons. If we denote by F, E and V the number faces, edges and vertices of a convex polyhedron, respectively, then the well-known Euler formula states [1]:

$$F - E + V = 2$$

Let us denote by F_k and V_k the number of k-gons and k-valent vertices (vertices of degree k). Then

$$F = \sum_{3\leq k} F_k, \; V = \sum_{3\leq k} V_k \quad \text{and} \quad 2E = \sum_{3\leq k} kF_k = \sum_{3\leq k} kV_k$$

By multiplying the Euler formula by six and using the above relations we obtain

$$(6F - 2E) + 2(3V - 2E) = 12 \qquad (1)$$

$$6\sum_{3\leq k} F_i - \sum_{3\leq k} kF_k + 2(\sum_{3\leq k} 3V_k - \sum_{3\leq k} kV_k) = 12 \qquad (2)$$

$$\sum_{3\leq k} (6-k)F_k + 2\sum_{3\leq k} (3-k)V_k = 12 \qquad (3)$$

In a similar manner we obtain

$$\sum_{3\leq k} (4-k)(F_k + V_k) = 8 \qquad (4)$$

Relations (3) and (4) state the necessary conditions for the existence of a convex polyhedron. (Note, that in (3) there is no restriction on the number of hexagons and 3-valent vertices, and in (4) there is no restriction on the number of quadrangles and 4-valent vertices.) It follows

$$3F_3 + 2V_4 + F_5 \geq 12 \quad \text{and} \quad F_3 + V_3 \geq 8$$

These two relations imply that the degree of the faces and the vertices of a regular polyhedron can not exceed five (a *regular polyhedron* is a polyhedron whose faces are regular polygons of the same type which are assembled in the same way around each vertex). Moreover, if the faces are triangles, then all the vertices must be of degree three. The above relations give the necessary conditions for the existence of convex regular polyhedrons; there are five of them and they are often called the Platonic solids: tetrahedron, hexahedron (cube), octahedron, dodecahedron, icosahedron.

The ball which had been used in the last 30 years of the 20-th century consisted of pentagons and hexagons only and all its vertices were of degree three. The above formulas imply that such a ball must consist of exactly 12 pentagons but the number of hexagons is not given. Since this condition is necessary but not sufficient, it is natural to ask how many hexagons there can be. It follows from several papers published in the second half of the 20-th century that all the numbers except for the number one are possible (see [1], page 61). Figure 1 depicts the graph of a regular dodecahedron which in fact provides the basis for the soccer ball, as we shall see bellow.

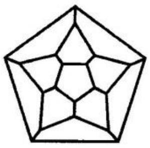

Figure 1. Dodecahedron

Given the planar graph of a polyhedron, it is possible to construct a new graph consisting of the same number of k-gons and new hexagons. This is illustrated on Figure 2. Dotted lines depict the original graph (the cube). On the left image the new graph is obtained by replacing all the original vertices by hexagons and the graph on the right is obtained by replacing all the original edges by hexagons. All the vertices of the new graphs are of degree three.

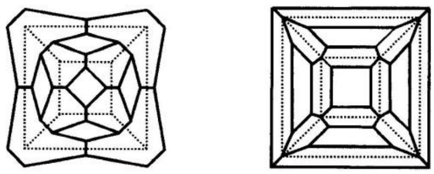

Figure 2. Transformations

Let us return to the soccer ball. It is constructed from the dodecahedron by replacing all the vertices by hexagons. Since the dodecahedron has 20 vertices, the soccer ball consists of 12 pentagons and 20 hexagons. By replacing the vertices by hexagons one more time, we obtain a ball consisting of 12 pentagons and 70 hexagons. By successive repetition we can obtain polyhedra consisting of 12 pentagons and a large number of hexagons.

By applying the second transformation on the dodecahedron, i.e. by replacing the edges by hexagons, we obtain a soccer ball consisting of 12 pentagons and 30 hexagons. If we apply this transformation on the cube, we obtain the ball used in volleyball. (Note: This ball can also be obtained from the octahedron by cutting-off the vertices.) Such balls were also used in football in the twenties of the last century.

Until the end of the 20th century, the ball was stitched together from pieces of leather. From practical reasons at most three pieces were stitched together in one point.

The first soccer balls in the 19th century were made of two digons stitched along the edges (such balls are still used in rugby). There were two vertices of degree two on the opposing sides and the ball was harder in these spots. All the balls used later were made so that at most three pieces of leather meet in one spot. Hence the balls were made of two 8-gons and 8 quadrangles, or other n-sided prism. Such balls were graph isomorphic with an n-sided polyhedron for $n = 8, 10, 12$.

The balls which were used later arised from the cube in four different ways. This is depicted on Figure 3 where the thick lines on each of the four pictures enclose one face of the cube.

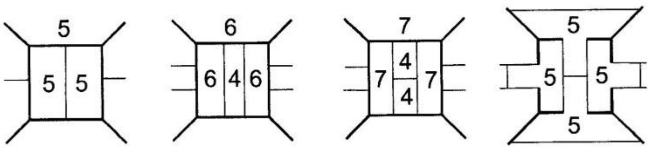

Figure 3. Constructions

Let us describe these constructions in detail.

1. We create a new vertex on each edge of the cube. We connect these vertices by six new edges so that on each face two new pentagons are created. Such a ball is graph isomorphic with the dodecahedron.

2. We create two new vertices on each edge and connect them by 12 edges so that on each face one new quadrangle and two new hexagons are created. Such a ball is graph isomorphic with the polyhedron obtained from the cube using the construction from Figure 2 on the right.

3. We start from the polyhedron constructed above - adding two vertices on each edge and 12 new edges. On each new edge we add a new vertex and connect each pair of the new vertices lying on the same face of the original cube by new a edge (thus creating the letter H). The faces of this polyhedron are 7-gons and quadrangles and their count is 24.

4. The polyhedron from point 3. contains 12 vertices incident with one quadrangle and two 7-gons. By removing the 6 edges connecting the pairs of these vertices we obtain a polyhedron which is graph isomorphic with the dodecahedron. Geometrically this polyhedron consists of 12 faces having the shape

pf the letter T. Such balls were used in the thirties of the 20th century. (Let us add that the valve was placed in one of the quadrangles, hence the two neighboring pentagons were transformed to hexagons.)

The above transformations can be done so that all the mirror planes of the cube are preserved. In such a case the resulting polyhedra have the same group of symmetries as the cube.

In the beginning of the 21st century synthetic materials replaced leather hence allowing for the creation of balls with various combinatorial structures. For the World Cup 2006 the ball called Teamgeist, containing six "biscuits " was used. The "biscuits" are quadrangles and the remaining faces are hexagons. This ball is graph isomorphic with the ball used in volleyball (depicted by full lines on Figure 2 on the right). For the world cup 2010 the ball called Jabulani was used. It is graph isomorphic with the tetrahedron whose vertices are replaced by triangles.

Observe the decorations on the soccer balls. The images "painted" on the ball usually reflect the structure of the faces. Some balls consisting of pentagons and hexagons have 5-starts painted on the pentagons; on others the hexagons are decorated by various ornaments. On some balls, the hexagons are replaced by triangles (in this case pairs of triangles "meet" in vertices of degree four). However, in each case there are two types of ornaments: 12 of them origin from pentagons and 20 from hexagons.

Finally let us consider metric properties of the soccer ball. Sometimes it is useful to know all the transformations which map the ball onto itself. These transformations are exactly the identities which map the dodecahedron onto intself. Carefully studying the soccer ball we can see that the dodecahedron, as well as the ideal ball, has 15 mirror planes. Each of these planes is given by a pair of "opposing" edges. By dividing the surface of the dodecahedron by the intersections of the planes with the surface, we obtain 120 elementary triangles. Since for each pair of triangles there exists exactly one identity mapping one onto the other, there are exactly 120 symmetries of the dodecahedron. Out of them 60 are direct and 60 opposite symmetries. However two identical shapes can not be identified in the 3-space. (More information on the symmetries of some polyhedra can be found in [2].)

Problem: In how many different (non-isomorphic) ways can we assign the numbers $1, 2, \dots, 12$ to the pentagons of the soccer ball?

Solution: If the ball was fixed on a base then this number would be equal to the number of permutations of a 12-element set, i.e. 12!. However, there are 60 ditect symmetries (rotations) the dodecahedron and hence also the ball onto itself. Therefore among the 12! balls with assigned numbers there are groups of 60 balls whose numbers are the same when the ball is rotated in the proper way. Hence the number of different assignments is $\frac{12!}{60}$.

Regular polyhedra have three groups of symmetries. It is well known that the regular tetrahedron (cube and octahedron, respectively dodecahedron and icosahedron) has four (9, resp. 15) mirror planes and it is preserved by 12 (24, resp. 60) rotations. We can use these facts to solve several problems.

We can formulate a problem similar to the one above for the cube with one to six dots on each side. The cube has 9 mirror planes; three of them are given by the centers of the faces and six by the opposing edges. These planes divide its surface into 48 elementary triangles, hence there exist 48 identities mapping the cube onto itself. Only a half of these symmetries is direct hence there are 24 different assignment of the numbers $\frac{6!}{24}$ to the faces.

(Note: Dies used in table-top games have the additional property that the sum of the numbers on the opposing faces is seven. There are only two non-isomorphic dies like this and they are mirror symmetric.)

Problem: In how many different ways can we assign the numbers $1, 2, \ldots, 8$ to the regular octahedron? How does this number change if we require the sum of the numbers on non-incident triangles to be the same?

The interested reader can formulate similar questions about different soccer balls.

References

[1] JUCOVIČ, E. *Konvexné mnohosteny.* Bratislava: Veda Bratislava, 1981.

[2] TRENKLER, M. On 4-valent 3-polytopes with a prescribed group of symmetries. *Graphs, Hypergraphs and Block systems, Zielona Gora*, 1976, 311–317.